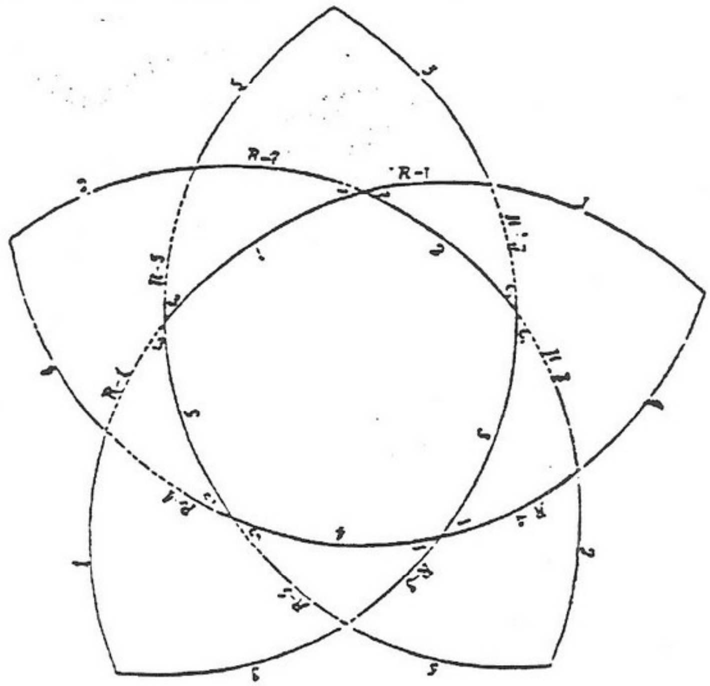
From chuck 5-12-07

DRAFT TRANSLATION OF PENTAGRAMA MIRIfICUM



[1.]

$$\begin{aligned}
\cos 1 &= \sin 2 \cdot \sin 5\\
\cos 2 &= \sin 3 \cdot \sin 1\\
\cos 3 &= \sin 4 \cdot \sin 2\\
\cos 4 &= \sin 5 \cdot \sin 3\\
\cos 5 &= \sin 1 \cdot \sin 4
\end{aligned}$$

$$\begin{aligned}
1 &= \cos 1 \cdot \tan 3 \cdot \tan 4\\
&= \cos 2 \cdot \tan 4 \cdot \tan 5\\
&= \cos 3 \cdot \tan 5 \cdot \tan 1\\
&= \cos 4 \cdot \tan 1 \cdot \tan 2\\
&= \cos 5 \cdot \tan 2 \cdot \tan 3
\end{aligned}$$

8. $-d1 = d2 \cdot \cos 4 + d5 \cdot \cos 3$

9. $-d2 = d3 \cdot \cos 5 + d1 \cdot \cos 4$

10. $-d3 = d4 \cdot \cos 1 + d2 \cdot \cos 5$

11. $-d4 = d5 \cdot \cos 2 + d3 \cdot \cos 1$

12. $-d5 = d1 \cdot \cos 3 + d4 \cdot \cos 2$

Eliminating d5 from 8 and 12 gives

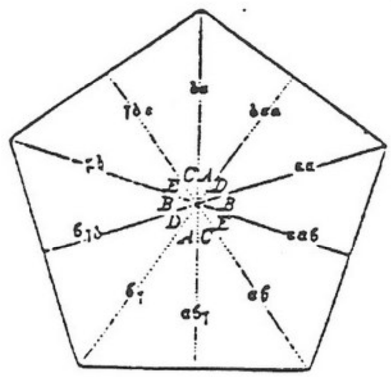
$$d1 . \sin 3^2 = -d2 . \cos 4 + d4 . \cos 2 . \cos 3$$

Whereas, eliminating d3 from 9 and 10

$d2.\sin 5^2 = -d1.\cos 4 + d4.\cos 1.\cos 5$

[2.]

The pentagon in the plane, which is formed by the central projection of a spherical one on any plane of your choosing, has the property, that normals drawn from a side to the opposing vertex intersect in one point (the eye point). At the same time the products of both pieces, where that common intersection point divides the normals, are always equal.



$\cos A = \alpha,\ \cos B = \beta,\ \cos C = \gamma,\ \cos D = \delta,\ \cos E = \epsilon$

Maintaining that whole complex numbers are expressed in each corner, there are five whole complex numbers p, p′, p′′, p′′′, p′′′′, and what is more,

$$p = a + bi,\ p' = a' + b'i,\ \text{etc.}$$

One sets

$$a'a''' + b'b''' = (1,3),\ a''a'''' = (2,4) \text{ and so forth}$$

and takes for the corners

(1,3)(2,4)p, (2,4)(3,0)p′, (3,0)(4,1)p′′, (4,1)(0,2)p′′′, (0,2)(1,3)p′′′′
= q q′, q′′, q′′′, q′′′′

therefore

$q=(a'a'''+b'b''')(a''a''''+b''b'''')a+(a'a'''+b'b''')(a''a''''+b''b'''')bi$
and so forth

Then

$q'-q = (ba'-ab')(2,4)(b'''-a'''i) = -(ba'-ab')(2,4)p'''i$ and so forth

The cut of qq' by oq''' has the complex number =

$$Q''' = \frac{(1,3)(2,4)(3,0)}{(a'''a'''+b'''b''')} p'''$$

The above mentioned product becomes

I pt.

$$= -(0,2)(1,3)(2,4)(3,0)(4,1)$$

$$Q''' - q = \frac{(1,3)(2,4)(ba'''-ab''')}{a'''a'''+b'''b'''}(b'''-a'''i)$$

$$q' - Q''' = \frac{(2,4)(3,0)(a'b'''-b'a''')}{a'''a'''+b'''b'''}(b'''-a'''i)$$

[3.]

The relations between the sides of the spherical pentagon are

Square of			Equations
Tangent	Secant	Cosecant	
α	$\gamma\delta$	$\gamma\delta/\alpha$	$1+\alpha=\gamma\delta$
β	$\delta\epsilon$	$\delta\epsilon/\beta$	$1+\beta=\delta\epsilon$
γ	$\epsilon\alpha$	$\epsilon\alpha/\gamma$	$1+\gamma=\epsilon\alpha$
δ	$\alpha\beta$	$\alpha\beta/\delta$	$1+\delta=\alpha\beta$
ϵ	$\beta\gamma$	$\beta\gamma/\epsilon$.	$1+\epsilon=\beta\gamma$

These equations are not independent, that is to say they are identical to:

$$\begin{aligned}(1+\gamma)(1+\beta-\delta\epsilon)-(1+\beta)(1+\gamma-\epsilon\alpha) &= \epsilon\{(1+\beta)a-(1+\gamma)\delta\} \\ &= \epsilon\{(\alpha\beta-\delta-1)-\gamma\delta-\alpha-1)\}\end{aligned}$$

and in a similar way the fifth is derived from the remaining three.

From two of the magnitudes $\alpha,\ \beta,\ \gamma,\ \delta,\ \epsilon$ follow the others

$$\beta=\frac{1+\alpha+\gamma}{\alpha\gamma},\ \beta=\frac{1+\delta}{\alpha},\ \gamma=\frac{1+\alpha}{\alpha\beta-1},\ \beta=\frac{1+\epsilon}{\alpha\epsilon-1}$$

$$\delta=\frac{1+\alpha}{\gamma},\ \gamma=\frac{1+\alpha}{\delta},\ \delta=\alpha\beta-1,\ \gamma=\alpha\epsilon-1$$

$$\epsilon=\frac{1+\gamma}{\alpha},\ \epsilon=\frac{1+\alpha+\delta}{\alpha\delta},\ \epsilon=\frac{1+\beta}{\alpha\beta-1},\ \delta=\frac{1+\alpha}{\alpha\epsilon-1}$$

	$\cos^2$	$\sin^2$	
$\alpha = 9$	1/10	9/10	71° 33′ 56″
$\beta=3/2\ (2/3)$	3/5	2/5	39 13 54
$\gamma = 2$	1/3	2/3	54 44 7
$\delta = 5$	1/6	5/6	65 54 19
$\epsilon = 1/3$	3/4	1/4	30 0 0

The beautiful equation

$$3+\alpha+\beta+\gamma+\delta+\epsilon=\alpha\beta\gamma\delta\epsilon=\sqrt{(1+\alpha)(1+\beta)(1+\gamma)(1+\delta)(1+\epsilon)}$$

The area of the spherical pentagons is 360° minus the sum of the sides. If one sets the sum = to S and

$$(1+i\sqrt{\alpha})(1+i\sqrt{\beta})(1+i\sqrt{\gamma})(1+\sqrt{\delta})(1+\sqrt{\epsilon})=A+Bi$$

which becomes

$$A=\alpha\beta\gamma\delta\epsilon\cdot\cos S$$
$$B=\alpha\beta\gamma\delta\epsilon\cdot\sin S$$

[4.]

$$G(\alpha x+\beta y+\gamma z)^2+G'(\alpha' x+\beta' y+\gamma' z)^2+G''(\alpha'' x+\beta'' y+\gamma'' z)^2$$
$$=Axx+Byy+Czz+2ayz+2bxz+2cxy$$

$$\begin{array}{lll} (A-G)\alpha+ & c\beta+ & b\gamma=0 \\ c\alpha+(B-G)\beta+ & & \alpha\gamma=0 \\ b\alpha+ & a+\beta+(C-G)\gamma=0 & \end{array}$$

$$\{\alpha(A-G)-bc\}a=\{b(B-G)-ac\}\beta=\{c(C-G)-ab\}\gamma$$

$$\frac{bc}{a(A-G)-bc}+\frac{ac}{b(B-G)-ac}+\frac{ab}{c(C-G)-ab}+1=0$$

$$(A-G)(B-G)(C-G)+2abc=aa(A-G)+bb(B-G)+cc(C-G)$$

$$\alpha\alpha=\frac{1}{1+\left(\frac{a(A-G)-bc}{b(B-G)-ac}\right)^2+\left(\frac{a(A-G)-bc}{c(C-G)-ab}\right)^2}$$

If one sets

$$\frac{bc}{a(A-x)-bc}+\frac{ac}{b(B-x)-ac}+\frac{ab}{c(C-x)-ab}+1=0$$

becoming indefinite

$$\{a(A-x)-bc\}\{b(B-x)-ac\}\{c(C-x)-ab\} = -abc(x-G)(x-G')(x-G'')$$

Differentiating and then setting x=G, it becomes

$$\{a(A-G)-bc\}\{b(B-G)-ac\}\{c(C-G)-ab\}$$
$$\times abc\left\{\frac{1}{(a(A-G)-bc)^2}+\frac{1}{(b(B-G)-ac)^2}+\frac{1}{(c(C-G)-ab)^2}\right\}$$
$$=abc(G-G')(G-G'')$$

from which

$$\alpha=\sqrt{\frac{[b(B-G)-ab][c(C-G)-ab]}{[a(A-G)-bc](G-G')(G-G'')}}$$

In conclusion an alteration is required if the magnitudes a, b, c vanish. The above first equations for G would then be identical.

[5.]

The equation for the points on the surface of the cone where the points (1), (2), (3), (4), (5) lie, when the apex of the cone is in the center of sphere and at the same time the origin of the of the coordinate system. The x axis goes through point (3), therefore the yz plane goes through (1) and (5), the y axis goes through (1).

	x	y	z
(3)	1	0	0
(4)	cos 1	0	sin 1
(5)	0	cos 3	sin 3
(1)	0	1	0
(2)	cos 5	cos 4	— cos 3 . sin 5

the equation

$$(z . \cos 1 - x . \sin 1)(z . \cos 3 - y . \sin 3) \cos 2 = zy$$

or

$$zz = xz\sqrt{\alpha} + yz\sqrt{\gamma} + \frac{1+\alpha+\gamma}{\sqrt{\alpha\gamma}} xy$$

Through a change in the coordinates the same thing can be brought into the Form

$$z'z'=Lx'x'+My'y'$$

solving the equation for

$$t(2t-1)^2=\alpha\beta\gamma\delta\epsilon(t-1)$$

which has a negative G and two positive roots (G', G") so it becomes

$$Gz'z'+G'x'x'+G''y'y'=0$$
$$GG'G''=-1/4\alpha\beta\delta\gamma\epsilon$$
$$(G-1)(G'-1)(G''-1)=-1/4$$
$$(2G-1)(2G'-1)2G''-1)=-\alpha\beta\delta\gamma\epsilon$$

for the above example

$$t(2t-1)^2=20(t-1)$$

roots -2.1973145, +1.06931815, +2.1279965

one sets

$$\alpha\delta\gamma\delta\epsilon = \omega \quad \text{und} \quad \sqrt{\frac{(18\omega\div 1)^2}{(3\omega\div 1)^3}} = \cos 3\phi = \frac{18w+1}{\sqrt{(3w+1)^3}}$$

which becomes

$$t = \tfrac{1}{2} - \tfrac{1}{2}\cos\tfrac{\phi}{?}.\sqrt{(3\omega\div 1)}$$

In proportion to the cubic equation

$$\frac{t-(2t-1)^3}{t-1}=\alpha\beta\gamma\delta\epsilon$$

(one conveniently solves this with Weidenbach's table where $1/yxx=\alpha\beta\gamma\delta\epsilon$ must be searched for as $1-x/1+x=y$ which then becomes $2t-1=1/x$.) One gets an overview from the following table:

t	$\alpha\delta\gamma\delta\epsilon$	t	$\alpha\delta\gamma\delta\epsilon$	t	$\alpha\delta\gamma\delta\epsilon$
∞	∞	÷1.9	÷16.6	÷1.0	∞
÷10	÷400.9	÷1.8	÷15.2	.....	negativ
÷ 9	÷325.1	÷1.7	÷14.0	—1.0	∞
÷ 8	÷257.1	÷1.6	÷12.9	—1.618034	÷ 11.0901699
÷ 7	÷197.2	÷1.5	÷12.0	—2	÷ 16.7
÷ 6	÷145.2	÷1.4	÷11.34	—3	÷ 36.7
÷ 5	÷101.2	÷1.309017	÷11.0901699	—4	÷ 64.8
÷ 4	÷ 65.3	÷1.3	÷11.09	—5	÷100.8
÷ 3	÷ 37.5	÷1.2	÷11.76	—∞	∞
÷ 2	÷ 18	÷1.1	÷15.84		

With that, therefore, three real roots are permitted, $\alpha\beta\gamma\delta\epsilon$ must be ≥ 11.0901699 or

$$\frac{11}{2}+\frac{\sqrt{125}}{2}$$

the limit values for t therefore are:

$$+1.309017=\frac{3+\sqrt{5}}{4}$$

and

$$-1.618034=-\frac{1+\sqrt{5}}{2}$$

[6.]

A,B,C;D,E are the vertices of the polygons.
a,b,c,d,e are the poles of the diagonals
0,1,2 are the three primary axes corresponding to the roots G, G′, G″ of the equation

$$\frac{t(2t-1)^2}{t-1}=(\text{tang}\,AB.\text{tang}\,BC.\text{tang}\,CD.\text{tang}\,DE.\text{tang}\,EA)^2$$

where G may be taken for the negative root, so that

$$Guu+G'u'u'+G''u'',u''=0$$

if u is defined as the coordinates of any one of the points A,B,C,D,E, then at the same time uu+u′u′+u″u″=1

The source of the primary theorem is contained in the two equations

$$\text{I.} \qquad \cos 0A.\cos 0b=-\frac{\text{tang}\,EA.(2G-1-\text{tang}\,AD^2)}{4(G-G')(G-G'')}$$

$$\text{II.} \qquad \cos 0A.\cos 0C=-\frac{2G-1-\frac{1}{\sin DE^2}}{4\cos BC.\cos AB(G-G')(G-G'')}$$

Equation I represents 30 equations, but II represents 15, allowing for every permutation of the axes and vertices. Still more elegant (in the initial description where α=tangCD^2 and so forth)

$$\text{I.}\quad \begin{cases} \cos 0C.\cos 0\mathrm{d} = \dfrac{1+\varepsilon-2G}{4(G'-G)(G''-G)}\cdot\sqrt{\varepsilon} = \mathfrak{A}.\sqrt{\varepsilon} \\ \cos 0D.\cos 0c = \dfrac{1+\varepsilon-2G}{4(G'-G)(G''-G)}\cdot\sqrt{\delta} = \mathfrak{A}.\sqrt{\delta} \end{cases}$$

$$\text{II.}\quad \cos 0B.\cos 0E = \frac{2\alpha+1-2\alpha G}{4(G'-G)(G''-G)}\cdot\sqrt{\delta\varepsilon} = \mathfrak{a}.\sqrt{\delta\varepsilon}$$

For ten equations of I there are only nine values because multiplying 5 gives the same result as multiplying by the five others. Therefore there must be among the ten magnitudes $\mathfrak{A}$. $\mathfrak{a}$. $\mathfrak{B}$. $\mathfrak{b}$ etc. four equations of condition which represent the most elegant

$$\delta\mathfrak{b}\mathfrak{c}\mathfrak{C} = \varepsilon\mathfrak{e}\mathfrak{b}\mathfrak{D}, \qquad \gamma\mathfrak{c}\mathfrak{b}\mathfrak{D} = \alpha\mathfrak{a}\mathfrak{e}\mathfrak{E} \text{ u. s. w. oder auch}$$
$$\delta\mathfrak{a}\mathfrak{A}\mathfrak{C} = \gamma\mathfrak{b}\mathfrak{B}\mathfrak{D}, \qquad \gamma\mathfrak{b}\mathfrak{B}\mathfrak{D} = \delta\mathfrak{e}\mathfrak{C}\mathfrak{E} \text{ u. s. w.}$$

But

$$\cos 0A^2 = \frac{\mathfrak{e}\mathfrak{a}\mathfrak{b}}{\mathfrak{c}\mathfrak{b}}.\alpha = \frac{\mathfrak{D}\mathfrak{b}\gamma}{\mathfrak{E}} = \frac{\mathfrak{E}\mathfrak{e}\delta}{\mathfrak{B}} \text{ u. s. w.,} \qquad \cos 0a^2 = \frac{\mathfrak{E}\mathfrak{D}}{\mathfrak{a}} \text{ u. s. w.}$$

[7.]

April 20, 1843. The eccentric anomalies φ, φ', φ'', φ''', φ'''' of the points A,B,C,D,E are combined by the equations (G is considered as negative)

$$\frac{\sin\frac{1}{2}(\varphi'''+\varphi'')}{\cos\frac{1}{2}(\varphi'''-\varphi'')} = \frac{G}{G''}.\sin\varphi, \qquad \frac{\cos\frac{1}{2}(\varphi'''+\varphi'')}{\cos\frac{1}{2}(\varphi'''-\varphi'')} = \frac{G}{G'}.\cos\varphi$$

$$\frac{\sin\frac{1}{2}(\varphi'+\varphi'''')}{\cos\frac{1}{2}(\varphi'-\varphi'''')} = \sqrt{\frac{G(G-1)}{G''(G''-1)}}.\sin\varphi = \frac{G(2G-1)}{G''(2G''-1)}\sin\varphi$$

$$\frac{\cos\frac{1}{2}(\varphi'+\varphi'''')}{\cos\frac{1}{2}(\varphi'-\varphi'''')} = \sqrt{\frac{G(G-1)}{G'(G'-1)}}.\cos\varphi = \frac{G(2G-1)}{G'(2G'-1)}\cos\varphi$$

The relationship among the angles φ^0, φ', φ'' are simplest to represent in the following way

$$\sqrt{\frac{G'}{G'-1}}=\xi,\ \sqrt{\frac{G''}{G''-1}}=\eta$$

becoming $\xi\eta=$

$$\frac{[illegible]}{[illegible]} = \frac{[illegible]}{[illegible]} = \frac{[illegible]}{[illegible]} = \frac{[illegible]}{[illegible]} = \frac{[illegible]}{[illegible]}$$

for $\xi\eta+1/\xi\eta-1$ gives a similar expression which is easily derived out of this.

In Zahlen	1	2	3	2	3	1
				log tang	log tang	log Δ
φ^0. =	50° 29′ 20″	80° 54′ 55″	49° 13′ 4″	0.79616	0.06418	9.81007
φ' =	92 56 38	55 49 27	15 16 12.5	0.16814	9.43617	9.182
φ'' =	162 8 14	83 48 52	59 41 16.5	0.96505	0.23313	9.97901
φ''' =	260 34 22	64 29 16.5	21 13 39	0.32127	9.58931	9.33515
φ'''' =	291 6 47	74 57 29	34 35 48	0.57068	9.83871	9.594558
				0.73196 = log $\xi\eta$		

[8.]

The $\varphi, \varphi', \varphi'', \varphi''', \varphi''''$ are nothing other than the amplitudes of five transcendental arguments which grow around 4/5 K (in the meaning of Jacobi p. 31) and where the modulus is k.

$$=\sqrt{\frac{\frac{1}{G'G'}-\frac{1}{G''G''}}{\frac{1}{G'G'}-\frac{1}{GG}}} = \sin\mu, \quad \cos\mu = \sqrt{\frac{\frac{1}{G''G''}-\frac{1}{GG}}{\frac{1}{G'G'}-\frac{1}{GG}}}$$

The transcendental argument itself taken as indefinite

$$= \int \frac{z\,dy - y\,dz}{\sqrt{(zz+yy)}} \cdot \sqrt{\frac{\frac{1}{G'G'} - \frac{1}{GG}}{\left(\frac{1}{G'G'} - \frac{1}{GG}\right)zz + \left(\frac{1}{G''G''} - \frac{1}{GG}\right)yy}}$$

Δ is used in the description of Jacobi so that

die drei Grössen	ebenso		proportional den Zahlen	oder
$\operatorname{tang} \frac{1}{2}(\varphi' - \varphi''')$	$\operatorname{tang} \frac{1}{2}(\varphi'' - \varphi^0)$	u. s. w.	$G(2G-1)\sqrt{\left(\frac{1}{G'G'} - \frac{1}{GG}\right)}$	$\operatorname{tang} \operatorname{am} \frac{1}{5}\bar{K}$
$\operatorname{tang} \frac{1}{2}(\varphi'' - \varphi')$	$\operatorname{tang} \frac{1}{2}(\varphi''' - \varphi'')$		$-G\sqrt{\left(\frac{1}{G'G'} - \frac{1}{GG}\right)}$	$\operatorname{tang} \operatorname{am} \frac{2}{5}\bar{K}$
$\Delta\varphi^0$	$\Delta\varphi'$		1	1

?,684095155

4,68345

1,795134656

1,79512

FURTHER FRAGMENTS ON THE PENTAGRAMA MIRIFICUM

The exponents of the rejuvenation of the primary axes of the central projection ellipse are

$$\sqrt{\frac{G'-1}{G'}} = \frac{2G'-1}{\sqrt{\alpha\beta\gamma\delta\varepsilon}} \quad \text{für die erste Axe} \quad \frac{1}{\sqrt{G'}},$$

$$\sqrt{\frac{G''-1}{G''}} = \frac{2G''-1}{\sqrt{\alpha\beta\gamma\delta\varepsilon}} \quad \text{für die zweite Axe} \quad \frac{1}{\sqrt{G''}},$$

or because

$$GG'G'' = -\tfrac{1}{4}\alpha\beta\gamma\delta\varepsilon,$$

$$(G-1)(G'-1)(G''-1) = -\tfrac{1}{4},$$

$$(2G-1)(2G'-1)(2G''-1) = -\alpha\beta\gamma\delta\varepsilon$$

the value of the rejuvenation of the projected axes = 1/1-2G

It is advantageous to bring in, next to the previous magnitudes G, G′,G″ also the roots of the equations

$$\frac{u^3+u}{uu-1} = \sqrt{\alpha\beta\gamma\delta\varepsilon}$$

ξ, η, ζ are the same so

$$\xi+\eta+\zeta = -\xi\eta\zeta = \sqrt{\alpha\beta\gamma\delta\varepsilon},$$
$$\xi\eta+\eta\zeta+\zeta\xi = 1,$$

$$G = \frac{\zeta\zeta}{\zeta\zeta-1} \quad \text{oder} \quad \zeta = \sqrt{\frac{G}{G-1}}.$$
$$G' = \frac{\xi\xi}{\xi\xi-1} \quad \text{oder} \quad \xi = \sqrt{\frac{G'}{G'-1}},$$
$$G'' = \frac{\eta\eta}{\eta\eta-1} \quad \text{oder} \quad \eta = \sqrt{\frac{G''}{G''-1}}.$$

The Coordinates referring to the inner pentagon (the spherical) are

$$\frac{\frac{x}{\xi}}{\sqrt{\frac{xx}{\xi\xi}+\frac{yy}{\eta\eta}+\frac{zz}{\zeta\zeta}}}, \quad \frac{\frac{y}{\eta}}{\sqrt{\frac{xx}{\xi\xi}+\frac{yy}{\eta\eta}+\frac{zz}{\zeta\zeta}}}, \quad \frac{\frac{z}{\zeta}}{\sqrt{\frac{xx}{\xi\xi}+\frac{yy}{\eta\eta}+\frac{zz}{\zeta\zeta}}};$$

the coordinates for the next inner one are

$$\frac{\frac{x}{\xi\xi}}{\sqrt{\frac{xx}{\xi^4}+\frac{yy}{\eta^4}+\frac{zz}{\zeta^4}}}, \quad \frac{\frac{y}{\eta\eta}}{\sqrt{\frac{xx}{\xi^4}+\frac{yy}{\eta^4}+\frac{zz}{\zeta^4}}}, \quad \frac{\frac{z}{\zeta\zeta}}{\sqrt{\frac{xx}{\xi^4}+\frac{yy}{\eta^4}+\frac{zz}{\zeta^4}}}$$

and so forth. On the other hand, the coordinates for the outer one are

$$\frac{\xi x}{\sqrt{\xi\xi xx+\eta\eta yy+\zeta\zeta zz}}, \quad \frac{\eta y}{\sqrt{\xi\xi xx+\eta\eta yy+\zeta\zeta zz}}, \quad \frac{\zeta z}{\sqrt{\xi\xi xx+\eta\eta yy+\zeta\zeta zz}},$$

For our example where $\alpha\beta\gamma\delta\epsilon = 20$ the number values are:

$\xi=3.9276268$ $\eta=1.3735071$ $\zeta=-0.9289980$

[10.]

λ, μ the coefficient of the rejuvenation (negative fraction).

$$\left.\begin{matrix} a, & a\lambda, & a\lambda\lambda, & a\lambda^3, & \cdots \\ b, & b\mu, & b\mu\mu, & b\mu^3, & \cdots \end{matrix}\right\}$$ successive primary axes of the projected ellipse

$$a\xi + b\eta i, \quad a\xi' + b\eta' i$$ of the five polygonal points (Star form)

λ, μ, ν are roots of the equation

$$\frac{\lambda\lambda+1}{\lambda^3-\lambda} = \frac{\mu\mu+1}{\mu^3-\mu} = \frac{\nu\nu+1}{\nu^3-\nu} = \sqrt{\alpha\beta\gamma\delta\epsilon} = \frac{1}{\omega}$$

$$\lambda + \mu + \nu = \lambda\mu\nu = \omega,$$

$$\lambda\mu + \mu\nu + \nu\lambda + 1 = 0,$$

$$\lambda\lambda + \mu\mu + \nu\nu = \omega\omega + 2.$$

$$\lambda = \text{tang}\, L, \qquad \mu = \text{tang}\, M, \qquad \nu = \text{tang}\, N,$$

$$L + M + N = 0.$$

$$L = -14°17'4'', \qquad M = -36°3'26'', \qquad N = 50°20'30''.$$

In our example

$$G' = \frac{1}{1-\lambda\lambda} = \frac{\cos L^2}{\cos 2L},$$

$$G'' = \frac{1}{1-\mu\mu} = \frac{\cos M^2}{\cos 2M},$$

$$G = \frac{1}{1-\nu\nu} = \frac{\cos N^2}{\cos 2N},$$

$$(1-\lambda\lambda)\xi\xi' + (1-\mu\mu)\eta\eta' + (1-\nu\nu) = 0,$$

$$\left(\lambda + \frac{1}{\lambda}\right)\xi\xi' + \left(\mu + \frac{1}{\mu}\right)\eta\eta' + \left(\nu + \frac{1}{\nu}\right) = 0,$$

$$\left(\lambda - \frac{1}{\lambda}\right)\xi\xi'' + \left(\mu - \frac{1}{\mu}\right)\eta\eta'' + \left(\nu - \frac{1}{\nu}\right) = 0,$$

$$\frac{\lambda\lambda+1}{\lambda\lambda}\,\xi\xi'' + \frac{\mu\mu+1}{\mu\mu}\,\eta\eta'' + \frac{\nu\nu+1}{\nu\nu} = 0,$$

$$\frac{\xi\xi'}{\sin 2L} + \frac{\eta\eta'}{\sin 2M} + \frac{1}{\sin 2N} = 0,$$

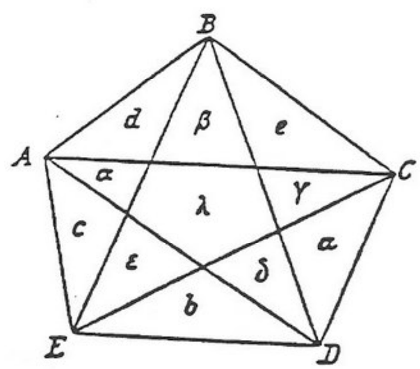
$$\frac{\xi\xi''}{\operatorname{tang} 2L} + \frac{\eta\eta''}{\operatorname{tang} 2M} + \frac{1}{\operatorname{tang} 2N} = 0.$$

The four points $\lambda\xi+\mu\eta i$, $\xi'+\eta' i$, $\xi''+\eta'' i$, $\lambda\xi'''+\mu\eta''' i$ lie in a straight line and just so for four other combinations of any four other points.

$$\frac{\lambda\lambda+1}{\lambda}(\xi''-\xi)\xi''' = \frac{\mu\mu+1}{\mu}(\eta-\eta'')\eta''',$$

$$\frac{\lambda\lambda+1}{\lambda\lambda}(\xi'-\xi'')\xi''' = \frac{\mu\mu+1}{\mu\mu}(\eta''-\eta)\eta'''.$$

[11.]



$$\beta+\varepsilon+\lambda = A,$$
$$\gamma+\alpha+\lambda = B,$$
$$\delta+\beta+\lambda = C,$$
$$\varepsilon+\gamma+\lambda = D,$$
$$\alpha+\delta+\lambda = E,$$

$$\gamma+\delta+\lambda = A',$$
$$\delta+\varepsilon+\lambda = B',$$
$$\varepsilon+\alpha+\lambda = C',$$
$$\alpha+\beta+\lambda = D',$$
$$\beta+\gamma+\lambda = E',$$

$$c+d+\alpha = \mathfrak{A},$$
$$d+e+\beta = \mathfrak{B},$$
$$e+a+\gamma = \mathfrak{C},$$
$$a+b+\delta = \mathfrak{D},$$
$$b+c+\varepsilon = \mathfrak{E},$$

$$a+b+c+d+e=s,$$
$$\alpha+\beta+\gamma+\delta+\varepsilon=\sigma,$$
$$A+B+C+D+E=S=A'+B'+C'+D'+E',$$
$$\mathfrak{A}+\mathfrak{B}+\mathfrak{C}+\mathfrak{D}+\mathfrak{E}=\mathfrak{S},$$
$$S=2\sigma+5\lambda=S',$$
$$\mathfrak{S}=\sigma+2s, \qquad \omega=s+\sigma+\lambda$$

Proportionalitäten:

$$aA=(e+\gamma)(b+\delta),$$
$$bB=(a+\delta)(c+\varepsilon),$$
$$cC=(b+\varepsilon)(d+\alpha),$$
$$dD=(c+\alpha)(e+\beta),$$
$$eE=(d+\beta)(a+\gamma),$$

all from the principle that the products taken away from one another are equal triangles (simply having common points or having no common sides) being reduced from quadrilaterals by the diagonals.

The general barycentric equation between four points e.g. (A) (B) (C) (D) is:

$$\triangle_A (A)+\triangle_B (B)+\triangle_C (C)+\triangle_D (D)=0,$$

where triangle $\triangle_A$ represents the triangle BCD, $\triangle_B$ the triangle CDA and so forth. (with consideration for the sign). In our case this becomes, if according to the above description one still sets:

$$A^{*}=\alpha+\gamma+\delta+a+\lambda, \quad \text{u. s. w.}$$
$$A''=\omega-\mathfrak{B}-\mathfrak{E}, \quad \text{u. s. w.},$$

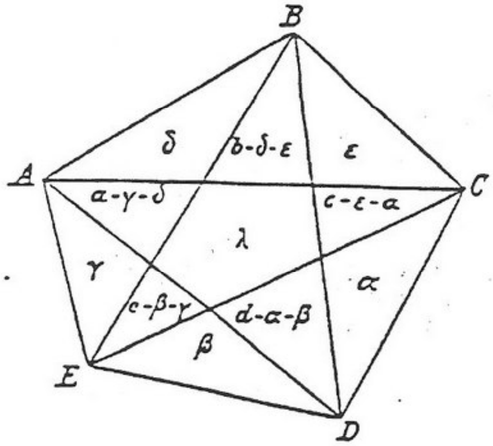
the type of equation between four points:

$$\mathfrak{C}(A)+D^{*}(C)=A^{*}(B)+\mathfrak{B}(D),$$

where A expresses the relevant corner point of the outer pentagons, or:

$$\mathfrak{C}(A)+(\omega-\mathfrak{E}-\mathfrak{D})(C)=(\omega-\mathfrak{B}-\mathfrak{E})(B)+\mathfrak{B}(D)=(\omega-\mathfrak{E})(e),$$

where (e) expresses the relevant point of the inner pentagons.



$$Ac\varepsilon + Cc(b-\varepsilon) = Bb(c-\varepsilon) + Db\varepsilon,$$
$$Bd\alpha + Dd(c-\alpha) = Cc(d-\alpha) + Ec\alpha,$$
$$Ce\beta + Ee(d-\beta) = Dd(e-\beta) + Ad\beta,$$
$$Da\gamma + Aa(e-\gamma) = Ee(a-\gamma) + Be\gamma,$$
$$Eb\delta + Bb(a-\delta) = Aa(b-\delta) + Ca\delta,$$

$$\lambda\alpha = (c-\alpha)(d-\alpha) - (e+b-\beta-\gamma-\delta-\varepsilon)\alpha,$$
$$\lambda\beta = (d-\beta)(e-\beta) - (a+c-\gamma-\delta-\varepsilon-\alpha)\beta,$$
$$\lambda\gamma = (e-\gamma)(a-\gamma) - (b+d-\delta-\varepsilon-\alpha-\beta)\gamma,$$
$$\lambda\delta = (a-\delta)(b-\delta) - (c+e-\varepsilon-\alpha-\beta-\gamma)\delta,$$
$$\lambda\varepsilon = (b-\varepsilon)(c-\varepsilon) - (d+a-\alpha-\beta-\gamma-\delta)\varepsilon,$$

or the next symbols

$$\lambda(0) = (8+9)(1+2) - 0(3+7)$$

or if

$$\alpha+\beta+\gamma+\delta+\varepsilon = \sigma, \qquad a+b+c+d+e = s$$

becomes set as

$$cd = (s+\lambda-\sigma-a)\alpha,$$
$$de = (s+\lambda-\sigma-b)\beta,$$
$$ea = (s+\lambda-\sigma-c)\gamma,$$
$$ab = (s+\lambda-\sigma-d)\delta,$$
$$bc = (s+\lambda-\sigma-e)\varepsilon.$$

therefore yields:

$$\lambda = \omega - s + \frac{cd}{\omega - a} + \frac{de}{\omega - b} + \frac{ea}{\omega - c} + \frac{ab}{\omega - d} + \frac{bc}{\omega - e},$$
$$= \frac{acd}{\omega\omega - \omega a} + \frac{bde}{\omega\omega - \omega b} + \frac{cea}{\omega\omega - \omega c} + \frac{dab}{\omega\omega - \omega d} + \frac{ebc}{\omega\omega - \omega e}.$$

consequently

$$ab + bc + cd + de + ea = S \text{ und } a\alpha + b\beta + c\gamma + d\delta + e\varepsilon = \Sigma$$

The area of the outer polygon is $=s+\lambda-\sigma=\omega$; therefore $S=\sigma\omega-\Sigma$.

REMARKS ON THE ELEVEN PENTAGRAM FRAGMENTS

The numerous developments and news about the Pentagramma Mirificum which are to be found in Gauss' estate, stem from very different time periods, and betray a notation [Gauss uses] which often changes. To this correspond the many different kinds of standpoints, from which the P. of Gauss is considered. What comes under consideration here in this respect, besides the eight fragments published in Volume III of the Collected Works, pp. 481 ff, are two conceptions which in three earlier-printed fragments, give the foundation [for all this]. A few further, unpublished elaborations contain in part reiterations, in part extensive numerical calculations, on the example again and again given by Gauss of $\alpha\beta\delta\gamma\epsilon = 20$.

In order to give an essential explication of the Fragments [9] to [11], it is necessary to reach back to the Fragments 1 to 8.

By Pentagramma, Gauss denotes a spherical pentagon, something done already by Neper [1] with his investigations into

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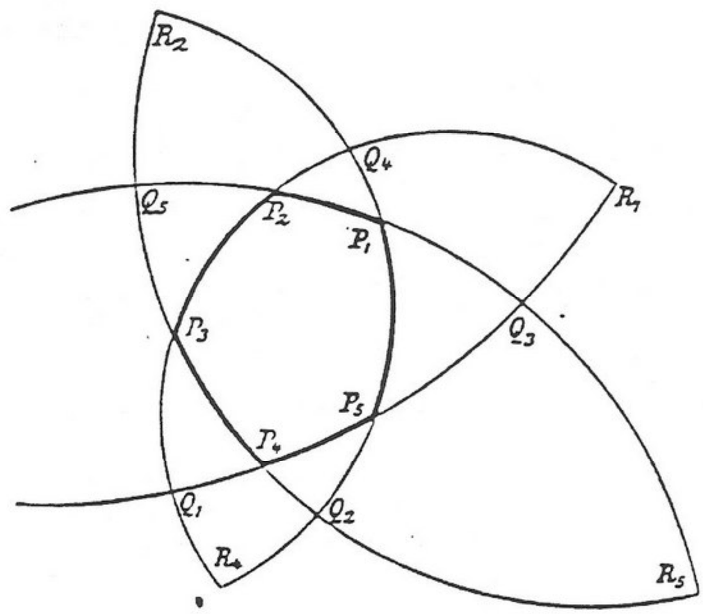
[1] Since the cited researches of Neper are relatively little known, so it is permitted here, a couple of remarks about the content of the same. They are contained in book II, Chapter IV of Neper's Mirifici Logarithmorum canonis desciptio (London 1619) and is the capstone in today's denoted Neperian rule'' denoted theorem: if h are the hypothenuse, a, b, like chords of a right angle spherical triangle, and if α, β be the angles lying opposite a and b, then it is true that for a, b, h, α, β are the five pieces that determine a right-angled spherical triangle then, the following relations are always true:

$$a'=\pi/2-h,\ b'=\pi/2-\beta,\ h'=\pi/2-b,\ \alpha'=\pi/2-a,\ \beta'=a$$

With that the transition from the first to the second triangle is one operation, which after being repeated five times, ends of itself, insofar as then you have gotten back to the original determining a,b,h, α, β. From this rule, Neper himself says, that it is readily at hand and comes out of the figure of a pentagon, just as Gauss also studied in the fragments we are discussing. Indeed we divide, in order to more closely study the figure described in the text, the triangle $P_1P_2Q_4$, thinking it to be upon the surface of a sphere, the determining arc segments $P_1Q_4=a$, $P_2Q_4=b$, $P_1P_2=h$ etc., then is precisely the neighbouring right angled triangle $P_2P_3Q_5$, which is appropriate to the determining fragments earlier denoted with a', b', h', α', β'. One can according to this directly go to a single right-angled spherical triangle as point of departure, and from it, by lengthening two sides for their complements, etc. to the remaining four angles/triangles of the figure, and with that attain to the pentagram. -

the right-angled spherical triangle, without re-entering angle, in which each single angle presents the pole of the opposite side, and in that according to this, the five diagonals are quadrants of the sphere. All this clearly to themselves polar pentagons make up a two-fold infinite continuum. The added figure presents us in stereographic projection a pentagram $P_1P_2P_3P_4P_5$.



The same is accompanied by two further two- and especially three-fold wound/enveloped pentagons $Q_1Q_2Q_3Q_4Q_5$ and $R_1R_2R_3R_4R_5$. As a consequence of the fundamental characteristic of the pentagramma, the angles of these two latter pentagons are totally right; if s_h are the length of the sides of the Pentagramma $P_1....P_5$ (and the radius of the sphere is set equal to 1) then one finds the sides of the other pentagons:

$$\widehat{P_h Q_{h+2}} = \frac{\pi}{2} - s_{h-1}, \qquad \widehat{P_h Q_{h-2}} = \frac{\pi}{2} - s_{h+1},$$

$$\widehat{Q_h R_{h+2}} = s_{h+1}, \qquad \widehat{Q_h R_{h-2}} = s_{h-1}.$$

Under the pentagrams there is to be found one especially regular. This has the length of side:

$$s=\arcsin\sqrt{\frac{-1+\sqrt{5}}{2}}$$

and lets itself be created from the spherical triangle of angles $2\pi/5$, $\pi/2$, $\pi/4$, through reproduction around the corner of the first angle.

What we just pointed to a general pentagram derived geometrical relation, allows one to put the fundamental formula of the right-angled spherical triangle in the form of a series of equations: this is carried out in the fragments 1 to 3. -

If we project the pentagram from the mid-point of the sphere, to a tangent plane, with a point of contact, O, lying in the Pentagram, then you get a plane pentagon, in which the five straight lines, OP_n give you the altitude. The plane pentagon has withal, moreover, two determining characteristics for itself:

1. the five heights /diagonals all run through one and the same point, O;
2. the individual heights are by O divided into two sections, whose product for all heights is the same, and (and indeed equal to the square of the radius of the sphere.

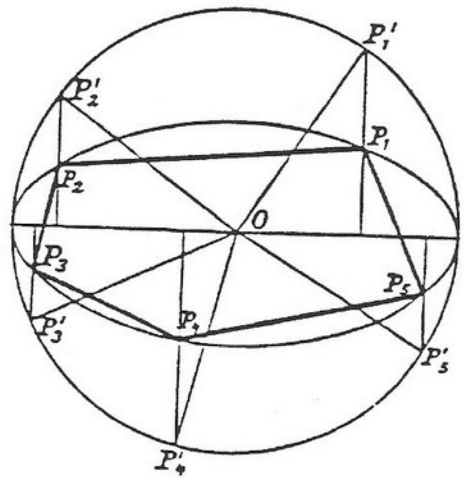
The projection plane makes Gauss now to the bearer of the complex numbers (cf. Fragment 2 and chooses in particular, O, as the zero point. The developments given later in Fragment 2, have the following sense. Upon five rays proceeding out of O, let five arbitrary points p, p′, ..., p′′′′′, be chosen; the shifting of these points as given by Gauss, each upon its ray, up to the points q, q′,...,q′′′′′, produces then in these last points, the angles of a pentagram of our kind. -

If M is the midpoint of the sphere, then it is possible through the 5 rays that are drawn to the corners of the pentragram P_n, MP_n, a uniquely determined cone of the second degree to construct. The last and namely the transformation formation of the same upon its main axes play in further unfolding of this by Gauss, a foundation-laying role. The cubic equation of this transformation of the main axes, which depends only of the product of the five tangents of the pentragram sides, will also be according to their numbered side in 5 be handled. If one wants for the rest, to prove the numerical relations of the fragments 5 and 6, it is best to just knot to the well-known more recent basic formulae for the transformation of the main axes of a cone. With the next thing chosen by Gauss, the coordinate system ,x,y, z, one can derived from those basic formula, the equations of the cited fragments, quite without calculation. -

Fricke is not a calculator

The development under 7 and 8, which carry the date 1843 April 20, are the foundation of the relation of the pentagram to the division into five of elliptical functions. It is without doubt, that Gauss in connection to this, under the stimulation of the treatise by Jacobi, ''about the application of the elliptical transcendental, to a familiar problem of elementary geometry *) has recognized. Jacobi studied the same pentagon, whose single a circle was inscribed, and at the same time is circumscribed by another circle, and points to the relation of this to elliptical functions.

In order to actually implement the transition from the pentagram to the Jacobi five-sided figure, let us project first from the mid point of the sphere to that tangential plane of the sphere, whose point of contact is with/of the diameter point of the sphere with the axis of the cone. The cone is cut by this projection plane into an ellipse, which is then inscribed into the projected pentagram. The affine transformations which are given in the figure below, produces in P'_1 $P'_2 \ldots P'_5$ a Jacobi pentagon.



The justness of this assertion stems from the agreement of the Gaussian formula in 7, and 8, with those of Jacobi a.a.O. developed equations. To prove the Gaussian relations one can proceed as follows:

Let us use in the projection plane the main axes of the ellipse as axes x, y and let the sphere radius equal 1, then the sphere midpoint lies perpendicularly upon the plane under O at a distance 1. The fundamental property of the pentagram, that namely angle $P_{h-1}MP_{h+1}=\pi/2$ is provided by the five equations:

$$(1) \qquad x_{h-1}x_{h+1} + y_{h-1}y_{h+1} + 1 = 0$$

when x_h and y_h, are the coordinates of P_h. The single pair, x_h, y_h, appears in two equations, through whose solution one finds

$$(2) \qquad x_h = \frac{y_{h+1} - y_{h-1}}{x_{h+1}y_{h-1} - x_{h-1}y_{h+1}}, \qquad y_h = \frac{x_{h-1} - x_{h+1}}{x_{h+1}y_{h-1} - x_{h-1}y_{h+1}}.$$

If now, as it is with Gauss, ω_h the eccentric anomaly of P_h, then the coordinates of P'_h are clearly $\cos \omega_h$, $\sin \omega_h$, and one finds as coordinates of P_h:

$$x_h = \sqrt{\frac{-G}{G'}}\cos\varphi_h, \qquad y_h = \sqrt{\frac{-G}{G''}}\sin\varphi_h,$$

G, G', and G'', in the sense as used by Gauss. Through the substitution of these values into equation 2, the first then will be found under 7 of the equations provided by Gauss:

$$\frac{\cos\frac{1}{2}(\varphi_{h+1} + \varphi_{h-1})}{\cos\frac{1}{2}(\varphi_{h+1} - \varphi_{h-1})} = \frac{G}{G'}\cos\varphi_h, \qquad \frac{\sin\frac{1}{2}(\varphi_{h+1} + \varphi_{h-1})}{\cos\frac{1}{2}(\varphi_{h+1} - \varphi_{h-1})} = \frac{G}{G''}\sin\varphi_h.$$

If one here eliminates ω_h, and writes after this k instead of k + 2, then it follows:

$$G'^2\cos^2\tfrac{1}{2}(\varphi_h + \varphi_{h+1}) + G''^2\sin^2\tfrac{1}{2}(\varphi_h + \varphi_{h+1}) = G^2\cos^2\tfrac{1}{2}(\varphi_h - \varphi_{h+1}),$$

$$G^2\cos(\varphi_{h+1} - \varphi_h) + (G''^2 - G'^2)\cos(\varphi_{h+1} + \varphi_h) = G'^2 + G''^2 - G^2.$$

Through developing the cosine and substituting the coordinates of the point P_h, it follows further:

$$(3)\quad (G^2 - G'^2 + G''^2)\, G' x_\nu x_{\nu+1} + (G^2 + G'^2 - G''^2)\, G'' y_\nu y_{\nu+1} + (-G^2 + G'^2 + G''^2)\, G = 0.$$

Now G, G′, G′′ are the roots of the equation

$$G(2G-1)^2 = \alpha\beta\gamma\delta\varepsilon(G-1).$$

There is to be found then according to this, an absolute member of the last equation:

$$(G^2 + G'^2 + G''^2 - 2G^2)\, G = \left(\frac{1+\alpha\beta\gamma\delta\varepsilon}{2} - 2G^2\right) G = \frac{\alpha\beta\gamma\delta\varepsilon}{2} \cdot \frac{1}{2G-1},$$

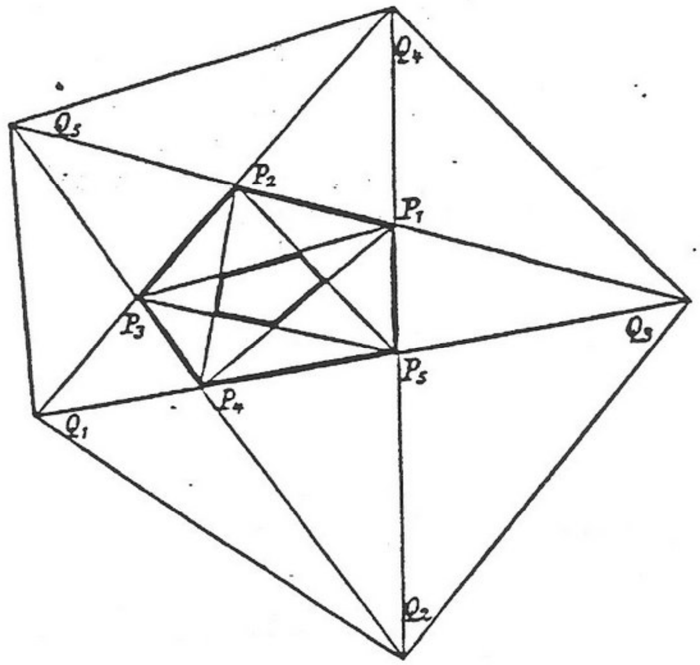
and one finds analogous expressions for the coefficients of the first two members of the equation (3), so that each equation is changed into:

$$(4)\qquad \frac{x_\nu x_{\nu+1}}{2G'-1} + \frac{y_\nu y_{\nu+1}}{2G''-1} + \frac{1}{2G-1} = 0.$$

To the extent that one deals with these equations the same was as (1), there will be found further those relations collected in fragment (7).

The assertions made here about the relation between Gauss and Jacobi, are put into a new light by these remarks, that each arbitrarily common pentagon, without re-entering angles, can become collinear in a Jacobi sense, and can thus also be transformed into a Gaussian pentagram. One can, namely, inscribe in the give pentagon an ellipse, and likewise circumscribe it with a determined conical cut. All that is needed, is to transform this pair of conical cuts into a pair of collinear circle cuts, which according to today's well-known methods, is not difficult.

This remark is also of importance for the elaborations carried out in the fragments ([9] to [11]. Gauss constructed here first by continuing the diagonal and the lengthening of sides, an chain endless in both [beginning of page 116] directions, of pentagons and recognizes, that this net is transformed into one-another circumscribing pentagons



though the collineation:

(5) $x'=(2G-1/2G'-1)x, \quad y'=(2G-1/2G''-1)y$

For making things clear, compare the figure provided; Gauss himself made ready drawings of this kind, in which the net of pentagons are carried through still much further. What in (5) stands on the right hand, the coefficients, are the ''exponents'' or ''coefficients of the rejuvenation'' which in [9] and [10] come into action. It lies very close by, to conceive the here-presented relationship of things, in the sense of modern theory of discontinuities substitution groups. The zero point, O, seems withal, as the inner boundary point of the pentragram net, and is one of the three boundary points of the from (5) originating cyclical collineation group.

As a result of the remarks sent off before, the same to be found with each pentagon [also] that has re-entering angles. This relationship is well-known in recent literature.

The justness of the Gaussian assertions can thus be confirmed: that to P_1P_2 .. P_5 first there be joined the external pentagon Q_1 Q_2 ... Q_5, is the same, which out of the so-called spherical pentagon originates upon projection. If Q_h has the coordinates x'_h, y'_h, so the value of angle $Q_hMP_{h+1} = \pi/2$ then the following is true:

$$x'_h x_{h+1} + y'_h y_{h+1} + 1 = 0$$

The comparison with the five equations (4) allows it to appear that for all indices k, the formula (5) appear to be valid.
p. 117

That Gauss was aware of the projective character of his development that we have just discussed, is made probable by the fragment (11). Gauss applies here the fundamental theorem of Moebius' barycentric calculus upon the figure of a straight-line pentagram; one will be able to, understand the formulae of Gauss, in the sense of this calculation, without difficulty. To the projective character of the latter, corresponding, here an arbitrary pentagon was attached. According to a letter to Schumacher of the 15 of May, 1843, Gauss had become acquainted with for the first time on the 14 of May of that year the (which appeared in 1827) the work of Moebius about the barycentric calculus, and probably, one might guess from stimulation of the task of finding the midpoint of a conical cut given by five points by construction. For the special pentagonal net, upon which the formula of the fragments [5] to [10] are related, that task should be with the problem of constructing the internal boundary point of a single net. For the rest, the development of the fragments 7 and 8 stem from the April 1843 and those of the Fragment 11, probably follows (immediately) to the one of the 14 of May 1843. After/according to this, we should not hesitate to assume, that the investigations undertaken in 9 and 10, were carried out in the weeks lying between those two dates.

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